Логарифмические функции — Интегралы

Здесь всюду x и a положительны (если нет особой оговорки).

610. $\int \ln x\,dx = x\ln x - x.$

610.01. $\int \ln(ax)\,dx = x\ln(ax) - x.$

610.1. $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$

610.2. $\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9}.$

610.3. $\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16}.$

610.9. $\int x^p\ln(ax)\,dx = \frac{x^{p+1}}{p+1}\ln(ax) - \frac{x^{p+1}}{(p+1)^2}$ $[p \neq -1].$

611.1. $\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}.$

611.11. $\int \frac{\ln(ax)}{x}\,dx = \frac{1}{2}\{\ln(ax)\}^2.$

611.2. $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}.$ **611.3.** $\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}.$

611.9. $\int \frac{\ln(ax)}{x^p}\,dx = -\frac{\ln(ax)}{(p-1)x^{p-1}} - \frac{1}{(p-1)^2x^{p-1}}$ $[p \neq 1].$

612. $\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x.$

612.1. $\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}.$

612.2. $\int x^2(\ln x)^2\,dx = \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27}.$

612.9. $\int x^p(\ln x)^2\,dx = \frac{x^{p+1}}{p+1}(\ln x)^2 - \frac{2x^{p+1}}{(p+1)^2}\ln x + \frac{2x^{p+1}}{(p+1)^3}$ $[p \neq -1].$

613.1. $\int \frac{(\ln x)^2\,dx}{x} = \frac{(\ln x)^3}{3}.$

613.2. $\int \frac{(\ln x)^2\,dx}{x^2} = -\frac{(\ln x)^2}{x} - \frac{2\ln x}{x} - \frac{2}{x}.$

613.3. $\int \frac{(\ln x)^2\,dx}{x^3} = -\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2}.$

613.9. $$\int \frac{(\ln x)^2\,dx}{x^p} = -\frac{(\ln x)^2}{(p-1)\,x^{p-1}} - \frac{2\ln x}{(p-1)^2\,x^{p-1}} - \frac{2}{(p-1)^3 x^{p-1}}$$
$[p \neq 1]$.

614. $$\int (\ln x)^3\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x.$$

615. $$\int (\ln x)^q dx = x(\ln x)^q - q\int (\ln x)^{q-1}\,dx \quad [q \neq -1].$$

616.1. $$\int \frac{(\ln x)^q\,dx}{x} = \frac{(\ln x)^{q+1}}{q+1} \qquad [q \neq -1].$$

616.2. $$\int x^p (\ln x)^q\,dx = \frac{x^{p+1}(\ln x)^q}{p+1} - \frac{q}{p+1}\int x^p (\ln x)^{q-1}\,dx$$
$[p,\ q \neq -1]$.

616.3. $$\int \frac{(\ln x)^q\,dx}{x^p} = \frac{-(\ln x)^q}{(p-1)\,x^{p-1}} + \frac{q}{p-1}\int \frac{(\ln x)^{q-1}\,dx}{x^p} \qquad [p,\ -q \neq 1].$$

617. $$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \frac{(\ln x)^2}{2\cdot 2!} + \frac{(\ln x)^3}{3\cdot 3!} + \dots$$

617.1. $$\int \frac{x\,dx}{\ln x} = \ln|\ln x| + 2\ln x + \frac{(2\ln x)^2}{2\cdot 2!} + \frac{(2\ln x)^3}{3\cdot 3!} + \dots$$

617.2. $$\int \frac{x^2 dx}{\ln x} = \ln|\ln x| + 3\ln x + \frac{(3\ln x)^2}{2\cdot 2!} + \frac{(3\ln x)^3}{3\cdot 3!} + \dots$$

617.9. $$\int \frac{x^p dx}{\ln x} = \ln|\ln x| + (p+1)\ln x + \frac{(p+1)^2(\ln x)^2}{2\cdot 2!} + \frac{(p+1)^3(\ln x)^3}{3\cdot 3!} + \dots$$
$\left[= \int \frac{e^y dy}{y},\ \text{где } y = (p+1)\ln x;\ \text{см. } \mathbf{568.1}\right]$.

618.1. $$\int \frac{dx}{x\ln x} = \ln|\ln x|.$$

618.2. $$\int \frac{dx}{x^2\ln x} = \ln|\ln x| - \ln x + \frac{(\ln x)^2}{2\cdot 2!} - \frac{(\ln x)^3}{3\cdot 3!} + \dots$$

618.3. $$\int \frac{dx}{x^3\ln x} = \ln|\ln x| - 2\ln x + \frac{(2\ln x)^2}{2\cdot 2!} - \frac{(2\ln x)^3}{3\cdot 3!} + \dots$$

618.9. $$\int \frac{dx}{x^p\ln x} = \ln|\ln x| - (p-1)\ln x + \frac{(p-1)^2(\ln x)^2}{2\cdot 2!} - \frac{(p-1)^3(\ln x)^3}{3\cdot 3!} + \dots$$

619.1. $$\int \frac{dx}{x(\ln x)^q} = \frac{-1}{(q-1)(\ln x)^{q-1}} \qquad [q \neq 1].$$

619.2. $$\int \frac{x^p\,dx}{(\ln x)^q} = \frac{-x^{p+1}}{(q-1)(\ln x)^{q-1}} + \frac{p+1}{q-1}\int \frac{x^p dx}{(\ln x)^{q-1}} \qquad [q \neq 1].$$

619.3. $$\int \frac{dx}{x^p (\ln x)^q} = \frac{-1}{x^{p-1}(q-1)(\ln x)^{q-1}} - \frac{p-1}{q-1}\int \frac{dx}{x^p (\ln x)^{q-1}}$$

$[q \neq 1]$.

620. $$\int \ln(a+bx)\,dx = \frac{a+bx}{b}\ln(a+bx) - x.$$

620.1. $$\int x \ln(a+bx)\,dx = \frac{b^2x^2-a^2}{2b^2}\ln(a+bx) + \frac{ax}{2b} - \frac{x^2}{4}.$$

621.1. $$\int \frac{\ln(a+bx)\,dx}{x} =$$

$$= (\ln a)\ln x + \frac{bx}{a} - \frac{b^2x^2}{2^2a^2} + \frac{b^3x^3}{3^2a^3} - \frac{b^4x^4}{4^2a^4} + \dots \qquad [b^2x^2 < a^2],$$

$$= \frac{(\ln bx)^2}{2} - \frac{a}{bx} + \frac{a^2}{2^2b^2x^2} - \frac{a^3}{3^2b^3x^3} + \frac{a^4}{4^2b^4x^4} - \dots \qquad [b^2x^2 > a^2].$$

621.2. $$\int \frac{\ln(a+bx)\,dx}{x^2} = \frac{b}{a}\ln x - \left(\frac{1}{x} + \frac{b}{a}\right)\ln(a+bx).$$

621.9. $$\int \frac{\ln(a+bx)\,dx}{x^p} = -\frac{\ln(a+bx)}{(p-1)x^{p-1}} + \int \frac{b\,dx}{(p-1)(a+bx)x^{p-1}}$$

$[p \neq 1]$. [См. 101—105.]

622. $$\int \frac{\ln x\,dx}{a+bx} = \frac{(\ln x)\ln(a+bx)}{b} - \int \frac{\ln(a+bx)\,dx}{bx}.$$ [См. 621.1.]

623. $$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\,\operatorname{arctg}\frac{x}{a}.$$

623.1. $$\int x\ln(x^2+a^2)\,dx = \frac{1}{2}\left[(x^2+a^2)\ln(x^2+a^2) - x^2\right].$$

623.2. $$\int x^2\ln(x^2+a^2)\,dx =$$

$$= \frac{1}{3}\left[x^3\ln(x^2+a^2) - \frac{2}{3}x^3 + 2xa^2 - 2a^3\operatorname{arctg}\frac{x}{a}\right].$$

623.3. $$\int x^3\ln(x^2+a^2)\,dx =$$

$$= \frac{1}{4}\left[(x^4-a^4)\ln(x^2+a^2) - \frac{x^4}{2} + x^2a^2\right].$$

623.4. $$\int x^4\ln(x^2+a^2)\,dx = \frac{1}{5}\left[x^5\ln(x^2+a^2) - \frac{2}{5}x^5 + \right.$$

$$\left. + \frac{2}{3}x^3a^2 - 2xa^4 + 2a^5\operatorname{arctg}\frac{x}{a}\right].$$

623.5. $$\int x^5\ln(x^2+a^2)\,dx =$$

$$= \frac{1}{6}\left[(x^6+a^6)\ln(x^2+a^2) - \frac{x^6}{3} + \frac{x^4a^2}{2} - x^2a^4\right].$$

623.6. $$\int x^6 \ln(x^2+a^2)\,dx = \frac{1}{7}\left[x^7 \ln(x^2+a^2) - \frac{2}{7}x^7 + \right.$$
$$\left. + \frac{2}{5}x^5a^2 - \frac{2}{3}x^3a^4 + 2xa^6 - 2a^7 \operatorname{arctg}\frac{x}{a}\right].$$

623.7. $$\int x^7 \ln(x^2+a^2)\,dx = \frac{1}{8}\left[(x^8-a^8)\ln(x^2+a^2) - \right.$$
$$\left. - \frac{x^8}{4} + \frac{x^6a^2}{3} - \frac{x^4a^4}{2} + x^2a^6\right].$$

624. $$\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2| - 2x + a\ln\left|\frac{x+a}{x-a}\right|.$$

624.1. $$\int x\ln|x^2-a^2|\,dx = \frac{1}{2}[(x^2-a^2)\ln|x^2-a^2| - x^2].$$

624.2. $$\int x^2\ln|x^2-a^2|\,dx = \frac{1}{3}\left[x^3\ln|x^2-a^2| - \frac{2}{3}x^3 - \right.$$
$$\left. - 2xa^2 + a^3\ln\left|\frac{x+a}{x-a}\right|\right].$$

624.3. $$\int x^3\ln|x^2-a^2|\,dx =$$
$$= \frac{1}{4}\left[(x^4-a^4)\ln|x^2-a^2| - \frac{x^4}{2} - x^2a^2\right].$$

624.4. $$\int x^4\ln|x^2-a^2|\,dx = \frac{1}{5}\left[x^5\ln|x^2-a^2| - \frac{2}{5}x^5 - \right.$$
$$\left. - \frac{2}{3}x^3a^2 - 2xa^4 + a^5\ln\left|\frac{x+a}{x-a}\right|\right].$$

624.5. $$\int x^5\ln|x^2-a^2|\,dx = \frac{1}{6}\left[(x^6-a^6)\ln|x^2-a^2| - \right.$$
$$\left. - \frac{x^6}{3} - \frac{x^4a^2}{2} - x^2a^4\right]$$

624.6. $$\int x^6\ln|x^2-a^2|\,dx = \frac{1}{7}\left[x^7\ln|x^2-a^2| - \frac{2}{7}x^7 - \right.$$
$$\left. - \frac{2}{5}x^5a^2 - \frac{2}{3}x^3a^4 - 2xa^6 + a^7\ln\left|\frac{x+a}{x-a}\right|\right].$$

624.7. $$\int x^7\ln|x^2-a^2|\,dx = \frac{1}{8}\left[(x^8-a^8)\ln|x^2-a^2| - \right.$$
$$\left. - \frac{x^8}{4} - \frac{x^6a^2}{3} - \frac{x^4a^4}{2} - x^2a^6\right].$$

Эти же выражения могут применяться к интегралам типа $\int x^p \ln(a^2-x^2)\,dx$.