Ряды

415.01. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad [x^2 < \infty].$$

415.02. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad [x^2 < \infty].$$

415.03. $$\operatorname{tg} x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$
$$\ldots + \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}x^{2n-1} + \ldots \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

[См. 45].

415.04. $$\operatorname{ctg} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \ldots$$
$$\ldots - \frac{2^{2n}B_n}{(2n)!}x^{2n-1} - \ldots \qquad [x^2 < \pi^2].$$

[См. 45.]

415.05. $$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \ldots$$
$$\ldots + \frac{E_n x^{2n}}{(2n)!} + \ldots \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

[См. 45].

415.06. $$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15\,120}x^5 + \frac{127}{604\,800}x^7 + \ldots$$
$$\ldots + \frac{2(2^{2n-1}-1)}{(2n)!}B_n x^{2n-1} + \ldots \qquad [x^2 < \pi^2].$$

[См. 45].

415.07. $$\sin(\theta + x) = \sin\theta + x\cos\theta - \frac{x^2\sin\theta}{2!} - \frac{x^3\cos\theta}{3!} + \frac{x^4\sin\theta}{4!} + \ldots$$

415.08. $$\cos(\theta + x) = \cos\theta - x\sin\theta - \frac{x^2\cos\theta}{2!} + \frac{x^3\sin\theta}{3!} + \frac{x^4\cos\theta}{4!} - \ldots$$

416.01. $$\frac{\pi}{4} = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots \qquad [0 < x < \pi].$$

416.02. Постоянная $$c = \frac{4c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\right) \qquad [0 < x > \pi].$$

416.03. $c=\frac{4c}{\pi}\left(\sin\frac{\pi x}{a}+\frac{1}{3}\sin\frac{3\pi x}{a}+\frac{1}{5}\sin\frac{5\pi x}{a}+\right.$ $\left.+\frac{1}{7}\sin\frac{7\pi x}{a}+\ldots\right)$ $[0<x<a]$.

416.04. $\frac{\pi}{4}=\cos x-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}-\frac{\cos 7x}{7}+\ldots$ $\left[-\frac{\pi}{2}<x<\frac{\pi}{2}\right]$.

416.05. Постоянная $c=\frac{4c}{\pi}\left(\cos x-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}-\frac{\cos 7x}{7}+\ldots\right)$ $\left[-\frac{\pi}{2}<x<\frac{\pi}{2}\right]$.

416.06. $c=\frac{4c}{\pi}\left(\cos\frac{\pi x}{a}-\frac{1}{3}\cos\frac{3\pi x}{a}+\frac{1}{5}\cos\frac{5\pi x}{a}-\frac{1}{7}\cos\frac{7\pi x}{a}+\ldots\right)$ $\left[-\frac{a}{2}<x<\frac{a}{2}\right]$.

416.07. $x=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots\right)$ $[-\pi<x<\pi]$.

416.08. $x=\pi-2\left(\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 4x}{4}+\ldots\right)$ $[0<x<2\pi]$.

416.09. $x=\frac{4}{\pi}\left(\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\ldots\right)$ $\left[-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}\right]$.

416.10. $x=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\ldots\right)$ $[0\leqslant x\leqslant\pi]$.

416.11. $x^2=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right)$ $[-\pi\leqslant x\leqslant\pi]$.

416.12. $x^2=\frac{\pi^2}{4}-\frac{8}{\pi}\left(\cos x-\frac{\cos 3x}{3^3}+\frac{\cos 5x}{5^3}-\right.$ $\left.-\frac{\cos 7x}{7^3}+\ldots\right)$ $\left[-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}\right]$.

416.13. $x^3-\pi^2x=$

$$=-12\left(\sin x-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\frac{\sin 4x}{4^3}+\ldots\right)$$ $[-\pi\leqslant x\leqslant\pi]$.

416.14. $\sin x = \frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} - \ldots\right)$
$\left[-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}\right].$

416.15. $\cos x = \frac{8}{\pi}\left\{\frac{\sin 2x}{1\cdot 3} + \frac{2}{3\cdot 5}\sin 4x + \frac{3}{5\cdot 7}\sin 6x + \ldots\right.$
$\left.\ldots + \frac{n}{(2n-1)(2n+1)}\sin 2nx + \ldots\right\}$
$[0 < x < \pi].$

416.16. $\sin ax = \frac{2\sin a\pi}{\pi}\left\{\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\right\},$
где a — не целое число $[-\pi < x < \pi].$

416.17. $\cos ax = \frac{2a\sin a\pi}{\pi}\left\{\frac{1}{2a^2} + \frac{\cos x}{1^2-a^2} - \frac{\cos 2x}{2^2-a^2} + \frac{\cos 3x}{3^2-a^2} - \ldots\right\},$
где a — не целое число. $[-\pi \leqslant x \leqslant \pi].$

417.1. $\frac{1}{1-2a\cos\theta+a^2} = 1 + \frac{1}{\sin\theta}(a\sin 2\theta + a^2\sin 3\theta +$
$+ a^3\sin 4\theta + \ldots)$ $[a^2 < 1].$

417.2. $\frac{1-a^2}{1-2a\cos\theta+a^2} = 1 + 2(a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots)$
$[a^2 < 1].$

417.3. $\frac{1-a\cos\theta}{1-2a\cos\theta+a^2} = 1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots$
$[a^2 < 1].$

417.4. $\frac{\sin\theta}{1-2a\cos\theta+a^2} = \sin\theta + a\sin 2\theta + a^2\sin 3\theta + \ldots$ $[a^2 < 1].$

418. $\ln(1-2a\cos\theta+a^2) =$
$= -2\left(a\cos\theta + \frac{a^2}{2}\cos 2\theta + \frac{a^3}{3}\cos 3\theta + \ldots\right)$ $[a^2 < 1],$
$= 2\ln|a| - 2\left(\frac{\cos\theta}{a} + \frac{\cos 2\theta}{2a^2} + \frac{\cos 3\theta}{3a^3} + \ldots\right)$ $[a^2 > 1].$

419.1. $e^{ax}\sin bx = \frac{rx\sin\theta}{1!} + \frac{r^2x^2\sin 2\theta}{2!} + \frac{r^3x^3\sin 3\theta}{3!} + \ldots,$
где $r = \sqrt{a^2+b^2}$, $a = r\cos\theta$ и $b = r\sin\theta$.

419.2. $$e^{ax}\cos bx = 1 + \frac{rx\cos\theta}{1!} + \frac{r^2x^2\cos 2\theta}{2!} + \frac{r^3x^3\cos 3\theta}{3!} + \ldots,$$
где r и θ те же, что и в **419.1.**

420.1. $$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha = \frac{\sin\frac{n+1}{2}\alpha\,\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

420.2. $$\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha = \frac{\cos\frac{n+1}{2}\alpha\,\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

420.3. $$\sin\alpha + \sin(\alpha+\delta) + \sin(\alpha+2\delta) + \ldots$$
$$\ldots + \sin\{\alpha + (n-1)\delta\} = \frac{\sin\left(\alpha + \frac{n-1}{2}\delta\right)\sin\frac{n\delta}{2}}{\sin\frac{\delta}{2}}.$$

420.4. $$\cos\alpha + \cos(\alpha+\delta) + \cos(\alpha+2\delta) + \ldots$$
$$\ldots + \cos\{\alpha + (n-1)\delta\} = \frac{\cos\left(\alpha + \frac{n-1}{2}\delta\right)\sin\frac{n\delta}{2}}{\sin\frac{\delta}{2}}.$$

421. Если $\sin\theta = x\sin(\theta+\alpha)$, то
$$\theta + r\pi = x\sin\alpha + \frac{1}{2}x^2\sin 2\alpha + \frac{1}{3}x^3\sin 3\alpha + \ldots \qquad [x^2 < 1],$$
где r — целое число.

422.1. $$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots \qquad [\theta^2 < \infty].$$

422.2. $$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots \qquad [\theta^2 < \infty].$$

См. также **818.1 — 818.4.**