Интегралы, содержащие $\sin x$ и $\cos x$

450.11. $$\int \sin x \cos x \, dx = \frac{\sin^2 x}{2} = -\frac{\cos^2 x}{2} + \text{const.},$$
$$= -\frac{\cos 2x}{4} + \text{const.}$$

450.12. $$\int \sin x \cos^2 x \, dx = -\frac{\cos^3 x}{3}.$$

450.13. $$\int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}.$$

450.19. $$\int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1}.$$

450.21. $$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3}.$$

450.22. $$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right).$$

450.23. $$\int \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x.$$

450.31. $$\int \sin^3 x \cos x \, dx = \frac{\sin^4 x}{4}.$$

450.81. $\int \sin^m x \cos x\, dx = \frac{\sin^{m+1} x}{m+1}$ $[m \neq -1]$.

(При $m = -1$ см. **453.11.**)

450.9. $\int \sin^m x \cos^n x\, dx =$

$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x\, dx,$

$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x\, dx$

$[m \neq -n$; см. **480.9**$]$. [См. также **461.**]

451.11. $\int \frac{dx}{\sin x \cos x} = \ln |\operatorname{tg} x|$.

451.12. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \left|\operatorname{tg} \frac{x}{2}\right|$.

451.13. $\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2\cos^2 x} + \ln |\operatorname{tg} x|$.

451.14. $\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \ln \left|\operatorname{tg} \frac{x}{2}\right|$.

451.15. $\int \frac{dx}{\sin x \cos^5 x} = \frac{1}{4\cos^4 x} + \frac{1}{2\cos^2 x} + \ln |\operatorname{tg} x|$.

451.19. $\int \frac{dx}{\sin x \cos^n x} = \frac{1}{(n-1)\cos^{n-1} x} + \int \frac{dx}{\sin x \cos^{n-2} x}$ $[n \neq 1]$.

451.21. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln \left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$.

451.22. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \operatorname{ctg} 2x = \operatorname{tg} x - \operatorname{ctg} x$.

451.23. $\int \frac{dx}{\sin^2 x \cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{\sin x} + \frac{3}{2} \ln \left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$.

451.24. $\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \operatorname{ctg} 2x$.

451.31. $\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2\sin^2 x} + \ln |\operatorname{tg} x|$.

451.32. $\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\cos x} - \frac{\cos x}{2\sin^2 x} + \frac{3}{2} \ln \left|\operatorname{tg} \frac{x}{2}\right|$.

451.33. $\int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2 \ln |\operatorname{tg} x|$.

451.41. $\int \frac{dx}{\sin^4 x \cos x} = \frac{3\cos^2 x - 4}{3\sin^3 x} + \ln \left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$.

451.91. $$\int \frac{dx}{\sin^m x \cos x} = -\frac{1}{(m-1)\sin^{m-1} x} + \int \frac{dx}{\sin^{m-2} x \cos x} \quad [m \neq 1].$$

451.92. $$\int \frac{dx}{\sin^n x \cos^n x} = 2^{n-1} \int \frac{d(2x)}{\sin^n (2x)}. \quad [\text{См. } \mathbf{432.}]$$

451.93. $$\int \frac{dx}{\sin^m x \cos^n x} =$$

$$= \frac{1}{(n-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2} x} \quad [n > 1],$$

$$-\frac{1}{(m-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cos^n x} \quad [m > 1].$$

452.11. $$\int \frac{\sin x\, dx}{\cos x} = \int \operatorname{tg} x\, dx = -\ln|\cos x| = \ln|\sec x|. \quad [\text{См. } \mathbf{480.1.}]$$

452.12. $$\int \frac{\sin x\, dx}{\cos^2 x} = \frac{1}{\cos x} = \sec x.$$

452.13. $$\int \frac{\sin x\, dx}{\cos^3 x} = \frac{1}{2\cos^2 x} = \frac{1}{2}\operatorname{tg}^2 x + \text{const}.$$

452.14. $$\int \frac{\sin x\, dx}{\cos^4 x} = \frac{1}{3\cos^3 x}.$$

452.19. $$\int \frac{\sin x\, dx}{\cos^n x} = \frac{1}{(n-1)\cos^{n-1} x} \quad [n \neq 1].$$

452.21. $$\int \frac{\sin^2 x\, dx}{\cos x} = -\sin x + \ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

452.22. $$\int \frac{\sin^2 x\, dx}{\cos^2 x} = \int \operatorname{tg}^2 x\, dx = \operatorname{tg} x - x. \quad [\text{См. } \mathbf{480.2.}]$$

452.23. $$\int \frac{\sin^2 x\, dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

452.24. $$\int \frac{\sin^2 x\, dx}{\cos^4 x} = \frac{1}{3}\operatorname{tg}^3 x.$$

452.29. $$\int \frac{\sin^2 x\, dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} x} \quad [n \neq 1].$$

452.31. $$\int \frac{\sin^3 x\, dx}{\cos x} = -\frac{\sin^2 x}{2} - \ln|\cos x|.$$

452.32. $$\int \frac{\sin^3 x\, dx}{\cos^2 x} = \cos x + \sec x.$$

452.33. $$\int \frac{\sin^3 x\, dx}{\cos^3 x} = \int \operatorname{tg}^3 x\, dx = \frac{1}{2}\operatorname{tg}^2 x + \ln|\cos x|. \quad [\text{См. } \mathbf{480.3.}]$$

452.34. $\displaystyle\int \frac{\sin^2 x\,dx}{\cos^4 x} = \frac{1}{3\cos^3 x} - \frac{1}{\cos x}.$

452.35. $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^5 x} = \frac{1}{4}\operatorname{tg}^4 x = \frac{1}{4\cos^4 x} - \frac{1}{2\cos^2 x} + \text{const}.$

452.39. $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^n x} = \frac{1}{(n-1)\cos^{n-1} x} - \frac{1}{(n-3)\cos^{n-3} x}$ $[n \neq 1 \text{ или } 3].$

452.41. $\displaystyle\int \frac{\sin^4 x\,dx}{\cos x} = -\frac{\sin^3 x}{3} - \sin x + \ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$

452.7. $\displaystyle\int \frac{\sin^{n-2} x\,dx}{\cos^n x} = \frac{\operatorname{tg}^{n-1} x}{n-1}$ $[n \neq 1]$

452.8. $\displaystyle\int \frac{\sin^n x\,dx}{\cos^n x} = \int \operatorname{tg}^n x\,dx = \frac{\operatorname{tg}^{n-1} x}{n-1} - \int \operatorname{tg}^{n-2} x\,dx$

$[n \neq 1;$ см. **480.9**$].$

452.9. $\displaystyle\int \frac{\sin^m x\,dx}{\cos^n x} =$

$$= \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\sin^m x\,dx}{\cos^{n-2} x} \qquad [n \neq 1],$$

$$= -\frac{\sin^{m-1} x}{(m-n)\cos^{n-1} x} + \frac{m-1}{m-n}\int \frac{\sin^{m-2} x\,dx}{\cos^n x} \qquad [m \neq n],$$

$$= \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1}\int \frac{\sin^{m-2} x\,dx}{\cos^{n-2} x} \qquad [n \neq 1].$$

453.11. $\displaystyle\int \frac{\cos x\,dx}{\sin x} = \int \operatorname{ctg} x\,dx = \ln|\sin x|.$ [См. **490.1.**]

453.12. $\displaystyle\int \frac{\cos x\,dx}{\sin^2 x} = -\frac{1}{\sin x} = -\csc x.$

453.13. $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x} = -\frac{1}{2\sin^2 x} = -\frac{\operatorname{ctg}^2 x}{2} + \text{const}.$

453.14. $\displaystyle\int \frac{\cos x\,dx}{\sin^4 x} = -\frac{1}{3\sin^3 x}.$

453.19. $\displaystyle\int \frac{\cos x\,dx}{\sin^n x} = -\frac{1}{(n-1)\sin^{n-1} x}$ $[n \neq 1].$

453.21. $\displaystyle\int \frac{\cos^2 x\,dx}{\sin x} = \cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

453.22. $\displaystyle\int \frac{\cos^2 x\,dx}{\sin^2 x} = \int \operatorname{ctg}^2 x\,dx = -\operatorname{ctg} x - x.$ [См. **490.2.**]

453.23. $\displaystyle\int \frac{\cos^2 x\,dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

453.24. $\displaystyle\int \frac{\cos^2 x\,dx}{\sin^4 x} = -\frac{1}{3}\operatorname{ctg}^3 x.$

453.29. $\int \frac{\cos^2 x\,dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} - \frac{1}{n-1}\int \frac{dx}{\sin^{n-2} x}$ $[n \neq 1]$.

453.31. $\int \frac{\cos^3 x\,dx}{\sin x} = \frac{\cos^2 x}{2} + \ln|\sin x|$.

453.32. $\int \frac{\cos^3 x\,dx}{\sin^2 x} = -\sin x - \csc x$.

453.33. $\int \frac{\cos^3 x\,dx}{\sin^3 x} = \int \operatorname{ctg}^3 x\,dx = -\frac{\operatorname{ctg}^2 x}{2} - \ln|\sin x|$. [См. **490.3.**]

453.34. $\int \frac{\cos^3 x\,dx}{\sin^4 x} = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}$.

453.35. $\int \frac{\cos^3 x\,dx}{\sin^5 x} = -\frac{1}{4}\operatorname{ctg}^4 x = \frac{1}{2\sin^2 x} - \frac{1}{4\sin^4 x} + \text{const}$.

453.39. $\int \frac{\cos^3 x\,dx}{\sin^n x} = \frac{1}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}$ $[n \neq 1$ или $3]$.

453.41. $\int \frac{\cos^4 x\,dx}{\sin x} = \frac{\cos^3 x}{3} + \cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|$.

453.7. $\int \frac{\cos^{n-2} x\,dx}{\sin^n x} = -\frac{\operatorname{ctg}^{n-1} x}{n-1}$ $[n \neq 1]$.

453.8. $\int \frac{\cos^n x\,dx}{\sin^n x} = \int \operatorname{ctg}^n x\,dx = -\frac{\operatorname{ctg}^{n-1} x}{n-1} - \int \operatorname{ctg}^{n-2} x\,dx$

$[n \neq 1$; см. **490.9**$]$.

453.9. $\int \frac{\cos^n x\,dx}{\sin^m x} =$

$$= -\frac{\cos^{n+1} x}{(m-1)\sin^{m-1} x} - \frac{n-m+2}{m-1}\int \frac{\cos^n x\,dx}{\sin^{m-2} x} \quad [m \neq 1],$$

$$= \frac{\cos^{n-1} x}{(n-m)\sin^{m-1} x} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} x\,dx}{\sin^m x} \quad [m \neq n],$$

$$= -\frac{\cos^{n-1} x}{(m-1)\sin^{m-1} x} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} x\,dx}{\sin^{m-2} x} \quad [m \neq 1].$$

454.01. $\int \frac{\sin x\,dx}{1+\cos x} = -\ln(1+\cos x)$.

454.02. $\int \frac{\sin x\,dx}{1-\cos x} = \ln(1-\cos x)$.

454.03. $\int \frac{\cos x\,dx}{1+\sin x} = \ln(1+\sin x)$.

454.04. $\int \frac{\cos x\,dx}{1-\sin x} = -\ln(1-\sin x)$.

454.05. $\int \frac{dx}{\sin x(1+\cos x)} = \frac{1}{2(1+\cos x)} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|$.

454.06. $\int \frac{dx}{\sin x\,(1-\cos x)} = -\frac{1}{2\,(1-\cos x)} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|$.

454.07. $\int \frac{dx}{\cos x\,(1+\sin x)} = -\frac{1}{2\,(1+\sin x)} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

454.08. $\int \frac{dx}{\cos x\,(1-\sin x)} = \frac{1}{2\,(1-\sin x)} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

454.09. $\int \frac{\sin x\,dx}{\cos x\,(1+\cos x)} = \ln\left|\frac{1+\cos x}{\cos x}\right|$.

454.10. $\int \frac{\sin x\,dx}{\cos x\,(1-\cos x)} = \ln\left|\frac{1-\cos x}{\cos x}\right|$.

454.11. $\int \frac{\cos x\,dx}{\sin x\,(1+\sin x)} = -\ln\left|\frac{1+\sin x}{\sin x}\right|$.

454.12. $\int \frac{\cos x\,dx}{\sin x\,(1-\sin x)} = -\ln\left|\frac{1-\sin x}{\sin x}\right|$.

454.13. $\int \frac{\sin x\,dx}{\cos x\,(1+\sin x)} = \frac{1}{2\,(1+\sin x)} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

454.14. $\int \frac{\sin x\,dx}{\cos x\,(1-\sin x)} = \frac{1}{2\,(1-\sin x)} - \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

454.15. $\int \frac{\cos x\,dx}{\sin x\,(1+\cos x)} = -\frac{1}{2\,(1+\cos x)} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|$.

454.16. $\int \frac{\cos x\,dx}{\sin x\,(1-\cos x)} = -\frac{1}{2\,(1-\cos x)} - \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|$.

455.01. $\int \frac{dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|$.

455.02. $\int \frac{dx}{\sin x-\cos x} = \frac{1}{\sqrt{2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}-\frac{\pi}{8}\right)\right|$.

455.03. $\int \frac{\sin x\,dx}{\sin x+\cos x} = \frac{x}{2} - \frac{1}{2}\ln|\sin x+\cos x|$. [См. **482.2** и **492.1.**]

455.04. $\int \frac{\sin x\,dx}{\sin x-\cos x} = \frac{x}{2} + \frac{1}{2}\ln|\sin x-\cos x|$. [См. **482.2** и **492.1.**]

455.05. $\int \frac{\cos x\,dx}{\sin x+\cos x} = \frac{x}{2} + \frac{1}{2}\ln|\sin x+\cos x|$. [См. **482.1** и **492.2.**]

455.06. $\int \frac{\cos x\,dx}{\sin x-\cos x} = -\frac{x}{2} + \frac{1}{2}\ln|\sin x-\cos x|$.

[См. **482.1** и **492.2.**]

455.07. $\int \frac{dx}{(\sin x+\cos x)^2} = \frac{1}{2}\operatorname{tg}\left(x-\frac{\pi}{4}\right)$.

455.08. $\int \frac{dx}{(\sin x-\cos x)^2}=\frac{1}{2}\operatorname{tg}\left(x+\frac{\pi}{4}\right)$.

455.09. $\int \frac{dx}{1+\cos x\pm\sin x}=\pm\ln\left|1\pm\operatorname{tg}\frac{x}{2}\right|$.

456.1. $\int \frac{dx}{b\cos x+c\sin x}=\frac{1}{r}\ln\left|\operatorname{tg}\frac{x+\theta}{2}\right|$,
где $r=\sqrt{b^2+c^2}$, $\sin\theta=b/r$, $\cos\theta=c/r$.
[См. **401.2** и **432.10.**]

456.2. $\int \frac{dx}{a+b\cos x+c\sin x}=\int \frac{d(x+\theta)}{a+r\sin(x+\theta)}$,
где r и θ даны в **456.1.** [См. **436.00.**]

460.1. $\int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg} x\right)$. [См. **436.5.**]

460.2. $\int \frac{dx}{a^2\cos^2 x-b^2\sin^2 x}=\frac{1}{2ab}\ln\left|\frac{b\operatorname{tg} x+a}{b\operatorname{tg} x-a}\right|$. [См. **436.7.**]

461. $\int \sin^m x\cos^n x\,dx$. Если одно из чисел m или n нечетное целое положительное, то следует сделать подстановку

$$\sin^2 x=1-\cos^2 x \quad \text{и} \quad \sin x\,dx=-d\cos x$$

или

$$\cos^2 x=1-\sin^2 x \quad \text{и} \quad \cos x\,dx=d\sin x.$$

Если оба числа m и n четные целые положительные, то следует сделать подстановки

$$\sin^2 x=\frac{1}{2}(1-\cos 2x), \quad \cos^2 x=\frac{1}{2}(1+\cos 2x)$$

и

$$\sin x\cos x=\frac{1}{2}\sin 2x.$$

Получив аналогичные выражения, но с аргументом $2x$ вместо x, преобразовать их далее подобным же образом.
См. также **450.9.**

465. $\int \sin mx\cos nx\,dx=-\frac{\cos(m-n)x}{2(m-n)}-\frac{\cos(m+n)x}{2(m+n)}$
$[m^2\neq n^2]$. [При $m^2=n^2$ см. **450.11.**]

470.1. $\int \frac{\cos x\,dx}{\sqrt{1+m^2\sin^2 x}}=\frac{1}{m}\ln(m\sin x+\sqrt{1+m^2\sin^2 x})$.

470.2. $\int \frac{\cos x\,dx}{\sqrt{1-m^2\sin^2 x}}=\frac{1}{m}\arcsin(m\sin x)$.

470.3. $$\int (\cos x)\sqrt{1+m^2\sin^2 x}\,dx = \frac{\sin x}{2}\sqrt{1+m^2\sin^2 x}+\frac{1}{2m}\ln\left(m\sin x+\sqrt{1+m^2\sin^2 x}\right).$$

470.4. $$\int (\cos x)\sqrt{1-m^2\sin^2 x}\,dx = \frac{\sin x}{2}\sqrt{1-m^2\sin^2 x}+\frac{1}{2m}\arcsin(m\sin x).$$

475.1. $$\int f(x,\ \sin x)\,dx = -\int f\left(\frac{\pi}{2}-y,\ \cos y\right)dy,$$

где $y=\frac{\pi}{2}-x.$

475.2. $$\int f(x,\ \cos x)\,dx = -\int f\left(\frac{\pi}{2}-y,\ \sin y\right)dy,$$

где $y=\frac{\pi}{2}-x.$